Electricity & magnetism-1

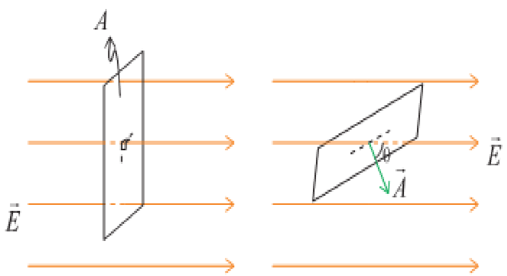
Electric flux and Gauss Law

Electric flux

Latin: flux = "to flow"

Graphically: Electric flux Φ_E represents the number of E-field lines crossing a surface.

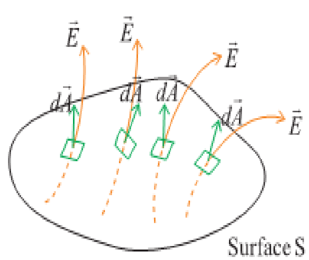
Mathematically:



$\Phi_E = EA$ $\qquad$ $\Phi_E = \vec{E} \cdot \vec{A}$

Reminder: Vector of the area $\vec{A}$ is perpendicular to the area A.

For non-uniform E-field & surface, direction of the area vector $\vec{A}$ is not uniform.



$d\vec{A}$ = Area vector for small area element dA

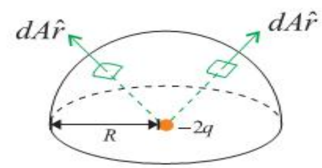
$\therefore$ Electric flux $\quad d\Phi_E = \vec{E} \cdot d\vec{A}$

Electric flux of $\vec{E}$ through surface S: $\boxed{\Phi_E = \int_S \vec{E} \cdot d\vec{A}}$

$$\int_S \; = \text{ Surface integral over surface S}$$
$$= \text{ Integration of integral over all area elements on surface S}$$

Example:

$S =$ hemisphere radius R



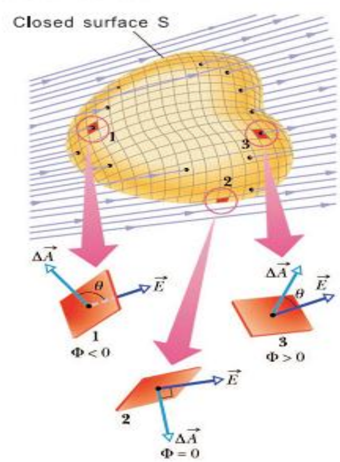
$$\boxed{\int_S dA = \text{Surface area of } S}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{-2q}{r^2}\,\hat{r} = \frac{-q}{2\pi\epsilon_0 R^2}\,\hat{r}$$

For a hemisphere, $d\vec{A} = dA\,\hat{r}$

$$\Phi_E = \int_S \frac{-q}{2\pi\epsilon_0 R^2}\,\hat{r} \cdot (dA\,\hat{r}) \qquad (\because \hat{r}\cdot\hat{r} =$$
$$= -\frac{q}{2\pi\epsilon_0 R^2} \underbrace{\int_S dA}_{2\pi R^2}$$
$$= \frac{-q}{\epsilon_0}$$

For a closed surface:



Recall: Direction of area vector $d\vec{A}$ goes *from inside to outside* of closed surface S.

Electric flux over closed surface S: $\boxed{\Phi_E = \oint_S \vec{E} \cdot d\vec{A}}$

$$\oint_S = \text{Surface integral over closed surface S}$$

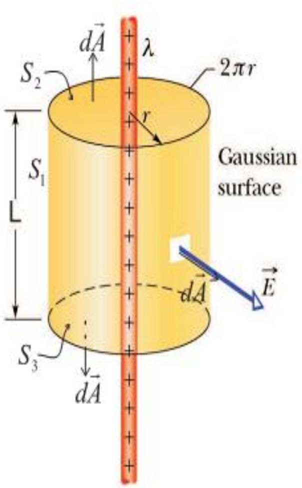
Gauss law

$$\boxed{\Phi_E = \oint_S \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}} \quad \text{for any closed surface S}$$

And q is the net electric charge enclosed in closed surface S.

- Gauss' Law is valid for *all charge distributions and all closed surfaces. (Gaussian surfaces)*
- Coulomb's Law can be derived from Gauss' Law.
- For system with high order of *symmetry*, E-field can be easily determined if we construct *Gaussian surfaces with the same symmetry* and applies Gauss' Law

For infinite line of charge



Linear charge density: λ
Cylindrical symmetry.
E-field directs radially outward from the rod.
Construct a Gaussian surface S in the shape of a **cylinder**, making up of a curved surface S_1, and the top and bottom circles S_2, S_3.

Gauss' Law: $\oint_S \vec{E}\cdot d\vec{A} = \frac{\text{Total charge}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$

As

$$\oint_S \vec{E}\cdot d\vec{A} = \underbrace{\int_{S_1} \vec{E}\cdot d\vec{A}}_{\vec{E}\parallel d\vec{A}} + \underbrace{\int_{S_2} \vec{E}\cdot d\vec{A} + \int_{S_3} \vec{E}\cdot d\vec{A}}_{=0\ \because \vec{E}\perp d\vec{A}}$$

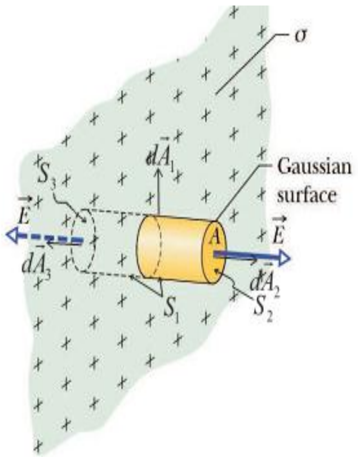
$$\therefore \quad E\underbrace{\int_{S_1} dA}_{\text{Total area of surface } S_1} = \frac{\lambda L}{\epsilon_0}$$

$$E(2\pi r L) = \frac{\lambda L}{\epsilon_0}$$

$$\therefore \boxed{E = \frac{\lambda}{2\pi\epsilon_0 r}} \qquad \text{(Compare with Chapter 2 note)}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\,\hat{r}$$

- # For infinite Sheet



Uniform surface charge density: σ

Planar symmetry.
E-field directs perpendicular to the sheet of charge.
Construct Gaussian surface S in the shape of a **cylinder (pill box)** of cross-sectional area A.

Gauss' Law: $\displaystyle \oint_S \vec{E}\cdot d\vec{A} = \frac{A\sigma}{\epsilon_0}$

$$\int_{S_1} \vec{E}\cdot d\vec{A} = 0 \quad \because \vec{E}\perp d\vec{A} \text{ over whole surface } S_1$$

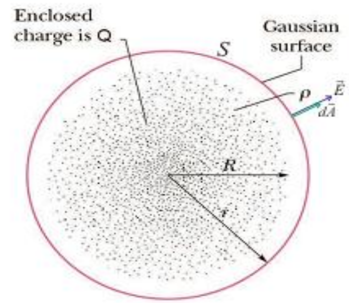
$$\int_{S_2} \vec{E}\cdot d\vec{A} + \int_{S_3} \vec{E}\cdot d\vec{A} = 2EA \quad (\vec{E}\parallel d\vec{A}_2, \vec{E}\parallel d\vec{A}_3)$$

Note: For S_2, both $\vec{E}$ and $d\vec{A}_2$ point up
For S_3, both $\vec{E}$ and $d\vec{A}_3$ point down

$$\therefore \quad 2EA = \frac{A\sigma}{\epsilon_0} \quad \Rightarrow \quad \boxed{E = \frac{\sigma}{2\epsilon_0}} \quad \text{(Compare with Chapter 2 note)}$$

- ## For uniform charge sphere

(a) For $r > R$:



Consider a spherical Gaussian surface S of radius r:

$$\vec{E} \parallel d\vec{A} \parallel \hat{r}$$

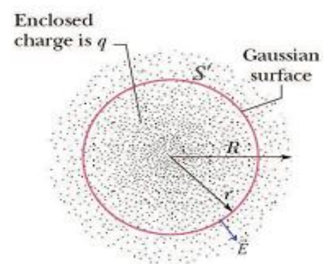
Gauss' Law: $\displaystyle \oint_S \vec{E}\cdot d\vec{A} = \frac{Q}{\epsilon_0}$

$$\oint_S E\cdot dA = \frac{Q}{\epsilon_0}$$

$$E\underbrace{\oint_S dA}_{\text{surface area of } S\, =\, 4\pi r^2} = \frac{Q}{\epsilon_0}$$

$$\therefore \quad \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{r}\,; \qquad \text{for } r > R$$

(b) For $r < R$:



Consider a spherical Gaussian surface S' of radius $r < R$, then total charge included q is *proportional to the volume included by* S'

$$\therefore \quad \frac{q}{Q} = \frac{\text{Volume enclosed by } S'}{\text{Total volume of sphere}}$$

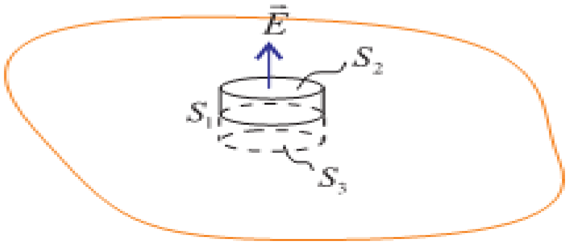
$$\frac{q}{Q} = \frac{4/3\,\pi r^3}{4/3\,\pi R^3} \quad \Rightarrow \quad q = \frac{r^3}{R^3}\,Q$$

Gauss' Law: $\displaystyle \oint_{S'} \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}$

$$E\underbrace{\oint_{S'} dA}_{\text{surface area of } S'\, =\, 4\pi r^2} = \frac{r^3}{R^3}\frac{1}{\epsilon_0}\cdot Q$$

$$\therefore \quad \vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{R^3}\,r\,\hat{r}\,; \qquad \text{for } r \le R$$

Gauss law and conductor



Cross-sectional area A

For *isolated* conductors, charges are free to move until *all charges lie outside the surface of the conductor.* Also, the *E-field at the surface of a conductor is perpendicular to its surface.* (Why?)

Consider Gaussian surface S of shape of cylinder:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{\sigma A}{\epsilon_0}$$

$$\text{BUT} \quad \int_{S_1} \vec{E} \cdot d\vec{A} = 0 \quad (\because \ \vec{E} \perp d\vec{A})$$

$$\int_{S_3} \vec{E} \cdot d\vec{A} = 0 \quad (\because \ \vec{E} = 0 \text{ inside conductor})$$

$$\begin{aligned} \int_{S_2} \vec{E} \cdot d\vec{A} &= E \underbrace{\int_{S_2} dA}_{\text{Area of } S_2} \quad (\because \ \vec{E} \parallel d\vec{A}) \\ &= EA \end{aligned}$$

$$\therefore \text{ Gauss' Law} \quad \Rightarrow \quad EA = \frac{\sigma A}{\epsilon_0}$$

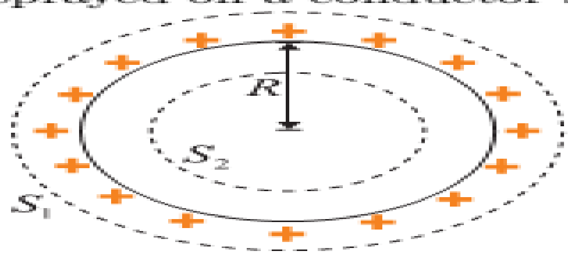
$$\therefore \ \boxed{\text{On conductor's surface} \quad E = \frac{\sigma}{\epsilon_0}}$$

BUT, there's no charge inside conductors.

$$\therefore \ \boxed{\text{Inside conductors} \quad E = 0} \quad \textit{Always!}$$

$\boxed{\text{Notice:} \quad \text{Surface charge density on a conductor's surface is } \textit{not uniform}.}$

I. Charge sprayed on a conductor sphere:



Total charge = Q

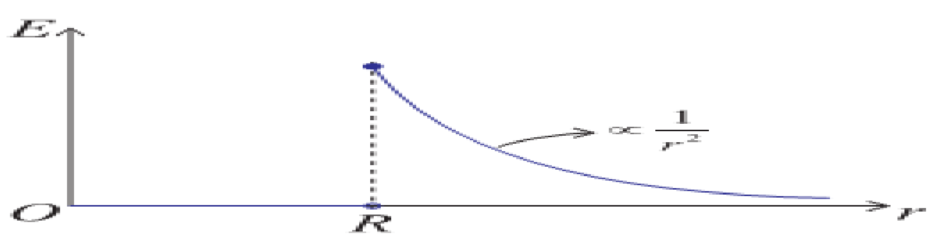
First, we know that charges
to the *surface* of conductors

(i) For $r < R$:
Consider Gaussian surface S_2

$$\oint_{S_2} \vec{E}\cdot d\vec{A} = 0 \quad (\because \text{ no charge inside })$$
$$\Rightarrow \quad E = 0 \quad \text{everywhere.}$$

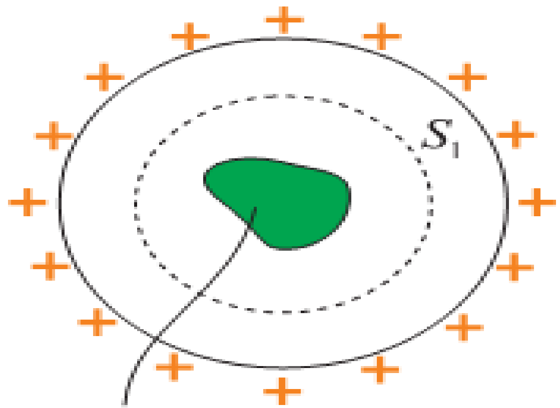
(ii) For $r \geq R$:
Consider Gaussian surface S_1:

$$\oint_{S_1} \vec{E}\cdot d\vec{A} = \frac{Q}{\epsilon_0}$$
$$E \underbrace{\oint_{S_1} d\vec{A}}_{4\pi r^2} = \frac{Q}{\epsilon_0} \qquad (\overbrace{\vec{E} \parallel d\vec{A}}^{\text{For a conductor}} \underbrace{\parallel \hat{r}}_{\text{Spherically symmetric}})$$
$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$



II. Conductor sphere with hole inside:

Continue…

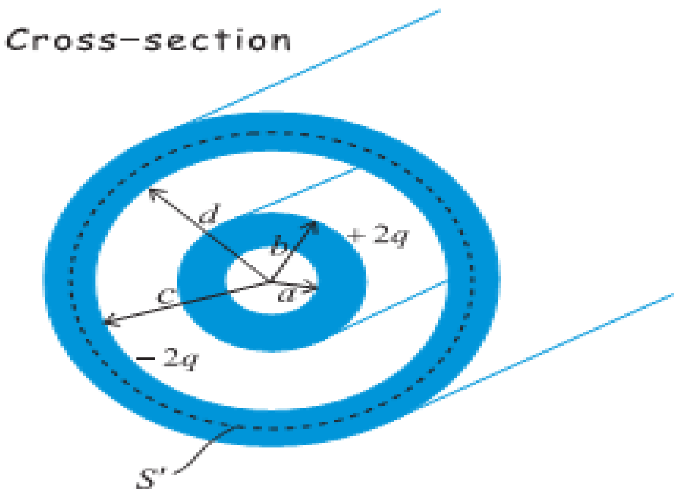


Consider Gaussian surface S_1:
charge included = 0

$\therefore$ E-field = 0 inside

The E-field is identical to the cas
solid conductor!!

III. A long hollow cylindrical conductor:



Example:
Inside hollow cylinder ($+2q$)

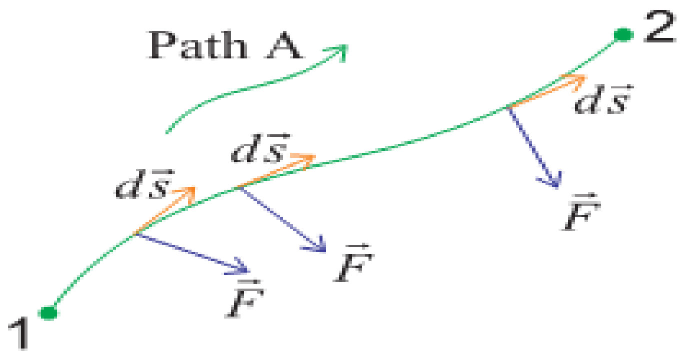
$$\begin{cases} \text{Inner radius} & \text{a} \\ \text{Outer radius} & \text{b} \end{cases}$$

Outside hollow cylinder ($-3q$)

$$\begin{cases} \text{Inner radius} & \text{c} \\ \text{Outer radius} & \text{d} \end{cases}$$

Electric potential energy

Electric force is a **conservative force**



Work done by the electric force $\vec{F}$ as a charge moves an infinitesimal distance d. along *Path A* $= dW$

Note: $d\vec{s}$ is in the *tangent* direction of the curve of *Path A*.

$$dW = \vec{F} \cdot d\vec{s}$$

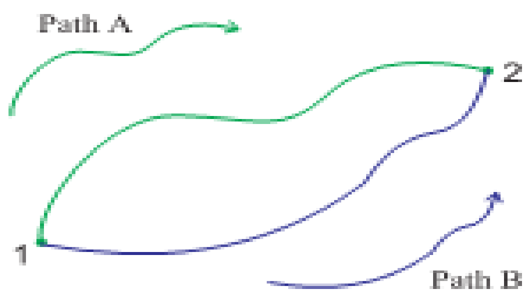
$\therefore$ Total work done W by force $\vec{F}$ in moving the particle from Point 1 to Point 2

$$\boxed{W = \int_{1\,\text{Path A}}^{2} \vec{F} \cdot d\vec{s}}$$

$$\int_{1\,\text{Path A}}^{2} = \textit{Path Integral}$$

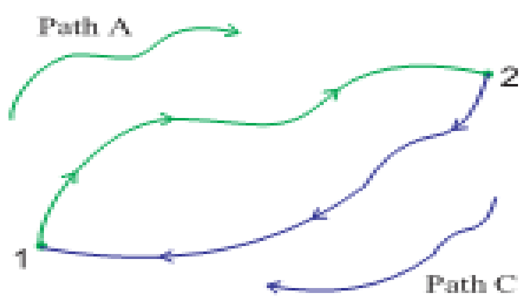
$$= \text{Integration over Path A from Point 1 to Point 2.}$$

Continue..



$\therefore$ For conservative forces,

$$\underset{\text{Path A}}{\int_1^2} \vec{F}\cdot d\vec{s} \quad = \quad \underset{\text{Path B}}{\int_1^2} \vec{F}\cdot d\vec{s}$$



Let's consider a path starting at point 1 to 2 through *Path A* and from 2 to 1 through *Path C*

$$\begin{aligned}
\text{Work done} \quad &= \quad \underset{\text{Path A}}{\int_1^2} \vec{F}\cdot d\vec{s} \quad + \quad \underset{\text{Path C}}{\int_2^1} \vec{F}\cdot d\vec{s} \\
&= \quad \underset{\text{Path A}}{\int_1^2} \vec{F}\cdot d\vec{s} \quad - \quad \underset{\textbf{Path B}}{\int_1^2} \vec{F}\cdot d\vec{s}
\end{aligned}$$

Continue….

DEFINITION: The work done by a **conservative force** on a particle when it *moves around a closed path returning to its initial position* is *zero.*

MATHEMATICALLY, $\vec{\nabla} \times \vec{F} = 0$ everywhere for conservative force $\vec{F}$

Conclusion: Since the work done by a conservative force $\vec{F}$ is *path-independent*, we can define a quantity, **potential energy**, that depends only on the *position* of the particle.

Convention: We define **potential energy** U such that

$$\boxed{dU = -W = -\int \vec{F} \cdot d\vec{s}}$$

$\therefore$ For particle moving from 1 to 2

$$\int_1^2 dU = U_2 - U_1 = -\int_1^2 \vec{F} \cdot d\vec{s}$$

where U_1, U_2 are **potential energy** at position 1, 2.

Exercise

27-1 Ans. $\Phi_E = (1800\,\mathrm{N/C})(3.2\times10^{-3}\mathrm{m})^2\cos(145°) = -7.8\times10^{-3}\mathrm{N\cdot m^2/C}.$

27-5 Ans. $\Phi_E = \dfrac{q}{\epsilon_0} = \dfrac{(1.84\,\mu\mathrm{C})}{(8.85\times10^{-12}\,\mathrm{C^2/N\cdot m^2})} = 2.08\times10^5\,\mathrm{N\cdot m^2/C}.$

27-9 Ans. There is no flux through the side of the cube. The flux of the top of the cube is $(-58\,\mathrm{N/C})(100\,\mathrm{m})^2 = -5.8\times10^5\mathrm{N\cdot m^2/C}$. The flux through the bottom of the cube is

$$(110\,\mathrm{N/C})(100\,\mathrm{m})^2 = 1.1\times10^6\mathrm{N\cdot m^2/C}.$$

The total flux is the sum, so the charge contained in the cube is

$$q = (8.85\times10^{-12}\mathrm{C^2/N\cdot m^2})(5.2\times10^5\mathrm{N\cdot m^2/C}) = 4.60\times10^{-6}\mathrm{C}.$$

27-12 Ans. $\lambda = 2\pi\epsilon_0 rE = 2\pi(8.85\times10^{-12}\mathrm{C^2/N\cdot m^2})(1.96\,\mathrm{m})(4.52\times10^4\mathrm{N/C}) = 4.93\times10^{-6}\mathrm{C/m}.$

All samples problem of ch#27 and exercise questions 27-19 and 27-21.